

ON CONTRACTIVELY COMPLEMENTED SUBSPACES OF SEPARABLE L_1 -PREDUALS

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ABSTRACT. It is shown that for an L_1 -predual space X and a countable linearly independent subset of $\text{ext}(B_{X^*})$ whose norm-closed linear span Y in X^* is w^* -closed, there exists a w^* -continuous contractive projection from X^* onto Y . This result combined with those of Pelczynski and Bourgain yields a simple proof of the Lazar-Lindenstrauss theorem that every separable L_1 -predual with non-separable dual contains a contractively complemented subspace isometric to $C(\Delta)$, the Banach space of functions continuous on the Cantor discontinuum Δ .

It is further shown that if X^* is isometric to ℓ_1 and (e_n^*) is a basis for X^* isometrically equivalent to the usual ℓ_1 -basis, then there exists a w^* -convergent subsequence $(e_{m_n}^*)$ of (e_n^*) such that the closed linear subspace of X^* generated by the sequence $(e_{m_{2n}}^* - e_{m_{2n-1}}^*)$ is the range of a w^* -continuous contractive projection in X^* . This yields a new proof of Zippin's result that c_0 is isometric to a contractively complemented subspace of X .

1. INTRODUCTION

A Banach space X is said to be an L_1 -predual provided its dual X^* is isometric to $L_1(\mu)$ for some measure space (Ω, Σ, μ) . Perhaps the most natural example of an L_1 -predual is $C(K)$, the Banach space of real-valued functions continuous on the compact Hausdorff space K , under the supremum norm. L_1 -preduals were the subject of an extensive study in the late 1960's and early 1970's. For a detailed survey of results on L_1 -preduals we refer to [13]. For the connection between L_1 -preduals and infinite-dimensional convexity we refer to the recent survey article [8].

For some time it was thought that every L_1 -predual is isomorphic to a $C(K)$ -space for suitable K , but the example given by Benyamini and Lindenstrauss [3] disproved this. The present paper is concerned with the existence of subspaces of a separable infinite-dimensional L_1 -predual X , isometric to $C(K)$ -spaces. It was proven by Zippin [27] that X contains a contractively complemented subspace isometric to c_0 . When X^* is non-separable, Lazar and Lindenstrauss [16] proved that X contains a contractively complemented subspace isometric to $C(\Delta)$, where Δ denotes the Cantor discontinuum. These results complement each other in the sense that neither of them implies the other.

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In the present paper we demonstrate a unified approach towards these results. Our method consists of establishing Theorem 3.2 which describes a technique for constructing a strictly increasing sequence (V_n) of finite-dimensional subspaces of X^* , where each V_n is isometric to some $\ell_1^{k_n}$, for which there exists an almost commuting sequence (P_n) of w^* -continuous contractive projections in X^* such that $\text{Im } P_n = V_n$, $n \in \mathbb{N}$. The proof of Theorem 3.2 is an elementary application of the principle of local reflexivity [17], [10]. We apply Theorem 3.2 in order to provide an alternative proof for the following result

Theorem 1.1. *Let X be an L_1 -predual and let K be a countable subset of $\text{ext}(B_{X^*})$ such that $K \cap (-K) = \emptyset$. Suppose that the norm-closed linear span Y of K is w^* -closed in X^* . Then there exists a w^* -continuous contractive projection from X^* onto Y .*

We remark that Theorem 1.1 is a consequence of the following result formulated by Lazar and Lindenstrauss as Corollary 1 in [16]:

Suppose that X is a Banach space so that X^ is isometric to $L_1(\mu)$. Let F be a face of B_{X^*} and denote by H the convex hull of $F \cup (-F)$. Assume that H is w^* -closed and metrizable. Then, there exists a w^* -continuous, affine, symmetric retraction of B_{X^*} onto H .*

Evidently, this result yields Theorem 1.1. However, the authors of [16] offer no proof of this result and moreover, via their preceding discussion, seem to require that the face F be w^* -closed. Note that if F is w^* -closed, so is H but the converse is not true in general. We also note that the proof of the aforementioned result that appears in [13] is false. Specifically, the map ϕ defined in the proof of the Corollary on page 224 of [13] is not convex.

Theorem 1.1 has applications in the study of ℓ_1 -preduals [1], [9]. As a consequence of Theorem 1.1 we obtain

Corollary 1.2. *Let X be a separable L_1 -predual and let K be a countable, w^* -compact subset of $\text{ext}(B_{X^*})$ such that $K \cap (-K) = \emptyset$. Then there exists a contractively complemented subspace of X isometric to $C(K)$.*

This corollary combined with the results of Pelczynski [22] and Bourgain [4] yields the next

Corollary 1.3. *Let X be a separable L_1 -predual such that X^* is non-separable. Then there exists a contractively complemented subspace of X isometric to $C(\Delta)$.*

This result was obtained in [16] with a different method. Their proof is based on a remarkable affine version of Michael's selection theorem [19] and makes use of a non-trivial result established in [14].

Another application of Theorem 3.2 is the following

Theorem 1.4. *Let X be a Banach space such that X^* is isometric to ℓ_1 , and let (e_n^*) be a basis for X^* isometrically equivalent to the usual ℓ_1 -basis. Then there exists a w^* -convergent subsequence $(e_{m_n}^*)$ of (e_n^*) such that the*

subspace generated by the sequence $(e_{m_{2n}}^* - e_{m_{2n-1}}^*)$ is the range of a w^* -continuous contractive projection in X^* .

Theorem 1.4 combined with Corollary 1.3 yields an alternative proof of Zippin's result [27]

Corollary 1.5. *Let X be a separable infinite-dimensional L_1 -predual. Then there exists a contractively complemented subspace of X isometric to c_0 .*

2. PRELIMINARIES

We shall make use of standard Banach space facts and terminology as may be found in [18]. In this section we review some of the necessary concepts. All Banach spaces under consideration will be over the field of real numbers. By the term *subspace* of a Banach space X we shall mean a closed linear subspace. We let B_X stand for the closed unit ball of X , while X^* denotes its topological dual. A subspace Y of X is said to be *complemented* if it is the range of a bounded linear projection $P: X \rightarrow X$. When $\|P\| = 1$, Y is a *contractively* complemented subspace of X .

ℓ_1 denotes the Banach space of absolutely summable sequences under the norm given by the sum of the absolute values of the coordinates. The usual ℓ_1 -basis is the Schauder basis of ℓ_1 consisting of sequences having exactly one coordinate equal to 1 and vanishing at the rest of the coordinates. ℓ_1^k , where $k \in \mathbb{N}$, is the k -dimensional subspace of ℓ_1 spanned by the first k members of the usual ℓ_1 -basis. A sequence (x_n) in some Banach space is *isometrically equivalent* to the usual ℓ_1 -basis, if $\|\sum_{i=1}^n a_i x_i\| = \sum_{i=1}^n |a_i|$, for all $n \in \mathbb{N}$ and scalar sequences $(a_i)_{i=1}^n$. A finite sequence $(x_i)_{i=1}^k$ in some Banach space is isometrically equivalent to the usual ℓ_1^k -basis, if $\|\sum_{i=1}^k a_i x_i\| = \sum_{i=1}^k |a_i|$, for all scalar sequences $(a_i)_{i=1}^k$.

c_0 stands for the Banach space of null sequences under the norm given by the supremum of the absolute values of the coordinates. ℓ_∞^n denotes the Banach space \mathbb{R}^n under the norm given by the maximum of the absolute values of the coordinates.

Given a measure space (Ω, Σ, μ) with μ positive, $L_1(\mu)$ denotes the Banach space of equivalence classes of absolutely integrable functions on Ω under the norm $\|f\| = \int_\Omega |f| d\mu$. $L_\infty(\mu)$ denotes the Banach space of equivalence classes of essentially bounded Σ -measurable functions on Ω under the norm $\text{ess sup}_{\omega \in \Omega} |f(\omega)|$.

An L_1 -predual is a Banach space X such that X^* is isometric to $L_1(\mu)$ for some measure space (Ω, Σ, μ) . According to a result of Pelczynski [21], Proposition 1.3, there exists another measure space (Ω', Σ', ν) with $L_1(\nu)$ isometric to $L_1(\mu)$ and such that $L_1(\nu)^*$ is canonically isometric to $L_\infty(\nu)$. Thus in the sequel, by an L_1 -predual we shall mean a Banach space X with X^* isometric to some $L_1(\mu)$ such that $L_1(\mu)^*$ is canonically isometric to $L_\infty(\mu)$.

Given a linear topological space V and $A \subset V$, we let $\text{co}(A)$ denote the convex hull of A . Let now K be a convex subset of V . A point $x \in K$ is

called an *extreme point*, if whenever y, z are in K and $x = ay + (1 - a)z$ for some $0 < a < 1$, then $x = y = z$. We let $\text{ext}(K)$ denote the set of extreme points of K . It is well known that for an L_1 -predual space X with X^* isometric to $L_1(\mu)$, $\text{ext}(B_{X^*})$ consists precisely of functions of the form $\sigma\chi_A/\mu(A)$, where $\sigma \in \{-1, 1\}$, A is an atom with $0 < \mu(A) < \infty$ and χ_A stands for the indicator function of A .

We next recall the important principle of local reflexivity [17], [10] (cf. also [26]).

Theorem 2.1. *Let X be a Banach space and let $E \subset X^{**}$ and $F \subset X^*$ be finite-dimensional subspaces. Given $\epsilon > 0$, there exists an invertible linear operator $T: E \rightarrow X$ such that $\|T\|\|T^{-1}\|TE\| \leq 1 + \epsilon$, $T|E \cap X = \text{id}$, and $f(TE) = e(f)$ for all $f \in F$ and $e \in E$.*

3. A CONSTRUCTION OF w^* -CONTINUOUS CONTRACTIVE PROJECTIONS

This section is devoted to the proof of Theorem 3.2 which provides a method of constructing w^* -continuous contractive projections onto certain finite-dimensional subspaces of $X^* = L_1(\mu)$, isometric to ℓ_1^k . Repeated applications of Theorem 3.2 will in turn enable us to construct a sequence (P_n) of almost commuting w^* -continuous contractive projections in X^* such that $(\text{Im } P_n)$ is strictly increasing and $\text{Im } P_n$ is isometric to some $\ell_1^{k_n}$, for all $n \in \mathbb{N}$. In order to construct w^* -continuous projections onto subspaces of X^* isometric to ℓ_1 , we shall make use of the following

Proposition 3.1. *Let X be a Banach space and let Y be a w^* -closed subspace of X^* . Assume that there exists a net $\{Y_\lambda\}_{\lambda \in \Lambda}$ of w^* -closed subspaces of Y with $Y_{\lambda_1} \subset Y_{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$ in Λ , and such that $\cup_{\lambda \in \Lambda} Y_\lambda$ is norm-dense in Y . Assume further that each Y_λ is the range of a w^* -continuous projection P_λ in X^* , so that $\sup_\lambda \|P_\lambda\| \leq M < \infty$, and $\lim_\lambda \sup_{\lambda \leq \mu} \|P_\lambda P_\mu - P_\lambda\| = 0$. Then there exists a w^* -continuous projection P from X^* onto Y with $\|P\| \leq M$.*

Proof. B_Y is w^* -compact. By Tychonoff's theorem we infer that $K = \prod_{x^* \in B_{X^*}} MB_Y$ is compact when endowed with the cartesian topology. We can now identify $\{P_\lambda\}_{\lambda \in \Lambda}$ with a net in K to obtain a sub-net $\{P_{\lambda'}\}_{\lambda' \in \Lambda'}$ of $\{P_\lambda\}_{\lambda \in \Lambda}$ which converges to an element P of K . Since Y is w^* -closed in X^* , P induces a bounded linear operator from X^* into Y , which we still denote by P . Clearly $Px^* = w^* - \lim_{\lambda' \in \Lambda'} P_{\lambda'}x^*$, for all $x^* \in X^*$ and thus $\|P\| \leq M$. Our assumptions yield that $Px^* = x^*$, for all $x^* \in \cup_{\lambda \in \Lambda} Y_\lambda$ and hence P is a projection onto Y .

We next demonstrate that P is w^* -continuous. By a classical result [12] it suffices to show that for every net (x_ν^*) in B_{X^*} such that $w^* - \lim_\nu x_\nu^* = \mathbf{0}$, we have that $w^* - \lim_\nu Px_\nu^* = \mathbf{0}$. Note that $\|Px_\nu^*\| \leq M$, for all ν , and let $y^* \in MB_Y$ be any w^* -cluster point of $(Px_\nu^*)_\nu$. We will show that $y^* = \mathbf{0}$. To this end set $\delta_\lambda = \sup_{\lambda \leq \mu} \|P_\lambda P_\mu - P_\lambda\|$. Then $\|P_\lambda P - P_\lambda\| \leq \delta_\lambda$, as P_λ is w^* -continuous. It follows that $\|P_\lambda Px_\nu^* - P_\lambda x_\nu^*\| \leq \delta_\lambda$, for all λ and ν ,

and thus as P_λ is w^* -continuous, we obtain that $\|P_\lambda y^*\| \leq \delta_\lambda$, for all $\lambda \in \Lambda$. Hence $Py^* = \mathbf{0}$. Because $y^* \in Y$ and P is a projection onto Y , we deduce that $y^* = \mathbf{0}$, completing the proof of the assertion. \square

We next pass to the key result which is related to Lemma 3.1 and Corollary 3.2 of [10].

Theorem 3.2. *Let X be an L_1 -predual and let V be a subspace of X^* isometric to ℓ_1^k . Let $(\delta_i)_{i=1}^n$ be a finite sequence of positive scalars and assume that there exist w^* -continuous linear operators $T_i: X^* \rightarrow V$, $\|T_i\| \leq 1$, $i \leq n$, as well as linear operators $R_i: V \rightarrow V$, $\|R_i\| \leq 1$, $i \leq n$, so that $\|R_i T_n - T_i\| < \delta_i$ for all $i \leq n$. Assume further that there exist collections of vectors $(f_j)_{j=1}^q \subset \text{ext}(B_{X^*})$ and $(v_j)_{j=1}^q \subset B_V$, with $(f_j)_{j=1}^q$ linearly independent, such that $\|R_i v_j - T_i f_j\| < \delta_i$ for all $i \leq n$ and $j \leq q$. Then there exists a w^* -continuous linear operator $T: X^* \rightarrow V$, $\|T\| \leq 1$, such that $\|R_i T - T_i\| < \delta_i$ for all $i \leq n$, and $T f_j = v_j$ for all $j \leq q$.*

The proof of this result will follow after establishing the next

Proposition 3.3. *Under the hypothesis of Theorem 3.2, for every $\epsilon > 0$ there exists a w^* -continuous linear operator $S: X^* \rightarrow V$, $\|S\| \leq 1$, such that $\|R_i S - T_i\| < \delta_i + \epsilon$ and $\|S f_j - v_j\| < \epsilon$ for all $i \leq n$ and $j \leq q$.*

Proof. Let $(e_l)_{l=1}^k$ be a basis for V isometrically equivalent to the usual ℓ_1^k -basis. Since T_i is w^* -continuous and $\|T_i\| \leq 1$, there exist vectors $(z_{il})_{l=1}^k$ in B_X such that $T_i x^* = \sum_{l=1}^k x^*(z_{il}) e_l$ for all $x^* \in X^*$. There also exist scalars (a_{ils}) , $i \leq n$, $l \leq k$, $s \leq k$, such that $R_i e_l = \sum_{s=1}^k a_{ils} e_s$ for all $i \leq n$ and $l \leq k$. Finally, there exist scalars (v_{jl}) , $j \leq q$, $l \leq k$, such that $v_j = \sum_{l=1}^k v_{jl} e_l$, $j \leq q$. Observe that for $x^* \in X^*$ and $i \leq n$ we have

$$\sum_{s=1}^k \left(\sum_{l=1}^k a_{ils} x^*(z_{nl}) \right) e_s = \sum_{l=1}^k x^*(z_{nl}) \sum_{s=1}^k a_{ils} e_s = R_i T_n x^*.$$

Hence, $\sum_{s=1}^k \left| \left(\sum_{l=1}^k a_{ils} x^*(z_{nl}) \right) - x^*(z_{is}) \right| = \|R_i T_n x^* - T_i x^*\| < \delta_i$, for all $x^* \in B_{X^*}$ and $i \leq n$. Thus

$$(1) \quad \left\| \sum_{s=1}^k \rho_s \left[\left(\sum_{l=1}^k a_{ils} z_{nl} \right) - z_{is} \right] \right\| < \delta_i,$$

for all choices of signs $(\rho_s)_{s=1}^k$ and all $i \leq n$.

Similarly, $R_i v_j = \sum_{s=1}^k \left(\sum_{l=1}^k a_{ils} v_{jl} \right) e_s$ for all $i \leq n$, $j \leq q$, and thus

$$(2) \quad \sum_{s=1}^k \left| \left(\sum_{l=1}^k a_{ils} v_{jl} \right) - f_j(z_{is}) \right| = \|R_i v_j - T_i f_j\| < \delta_i, j \leq q, i \leq n.$$

We first show that there exist vectors $(x_l^{**})_{l=1}^k$ in $B_{X^{**}} = B_{L_\infty(\mu)}$ such that $x_l^{**}(f_j) = v_{jl}$ for all $l \leq k$, $j \leq q$, and $\left\| \sum_{s=1}^k \rho_s \left[\left(\sum_{l=1}^k a_{ils} x_l^{**} \right) - z_{is} \right] \right\| \leq \delta_i$, for all $i \leq n$ and all choices of signs $(\rho_s)_{s=1}^k$. Indeed, let $\sigma_1, \dots, \sigma_q$ be signs

and let A_1, \dots, A_q be distinct atoms in (Ω, Σ, μ) such that $f_j = \sigma_j \chi_{A_j} / \mu(A_j)$ for all $j \leq q$. We can assume without loss of generality that the A_j 's are pairwise disjoint. We define $(x_l^{**})_{l=1}^k$ in $L_\infty(\mu)$ as follows:

$$x_l^{**}|_{A_j} = \sigma_j v_{jl}, j \leq q, \text{ while } x_l^{**}|_{\Omega \setminus \cup_{j \leq q} A_j} = z_{nl}|_{\Omega \setminus \cup_{j \leq q} A_j},$$

where we regard z_{nl} as an element of $X^{**} = L_\infty(\mu)$. Clearly, $\|x_l^{**}\| \leq 1$ and $x_l^{**}(f_j) = \int_{A_j} \sigma_j v_{jl} \sigma_j \chi_{A_j} / \mu(A_j) d\mu = v_{jl}$, for all $j \leq q$ and $l \leq k$.

Given signs ρ_1, \dots, ρ_k and $i \leq n$ we have that

$$(3) \quad \left\| \sum_{s=1}^k \rho_s \left[\left(\sum_{l=1}^k a_{ils} x_l^{**} \right) - z_{is} \right] |_{\Omega \setminus \cup_{j \leq q} A_j} \right\| \leq \left\| \sum_{s=1}^k \rho_s \left[\left(\sum_{l=1}^k a_{ils} z_{nl} \right) - z_{is} \right] \right\| < \delta_i \text{ by (1) .}$$

We next fix signs ρ_1, \dots, ρ_k , $i \leq n$ and $j \leq q$. We set

$$H_{ij}(t) = \sum_{s=1}^k \rho_s \left[\left(\sum_{l=1}^k a_{ils} \sigma_j v_{jl} \right) - z_{is}(t) \right], t \in A_j.$$

We claim that $|H_{ij}| < \delta_i$, μ -almost everywhere in A_j . Indeed, note first that $\int_{A_j} z_{is} d\mu = \sigma_j \mu(A_j) f_j(z_{is})$ for all $s \leq k$, and thus

$$(4) \quad \left| \int_{A_j} H_{ij} d\mu \right| = \left| \sum_{s=1}^k \rho_s \left[\left(\sum_{l=1}^k a_{ils} \sigma_j v_{jl} \mu(A_j) \right) - \sigma_j \mu(A_j) f_j(z_{is}) \right] \right| \leq \mu(A_j) \sum_{s=1}^k \left| \left(\sum_{l=1}^k a_{ils} v_{jl} \right) - f_j(z_{is}) \right| < \delta_i \mu(A_j), \text{ by (2) .}$$

On the other hand, setting $B_{ij} = \{t \in A_j : |H_{ij}(t)| \geq \delta_i\}$ and taking in account that A_j is an atom, we infer that $\mu(B_{ij}) = 0$. Indeed, otherwise, $\mu(B_{ij}) = \mu(A_j)$ and thus $|H_{ij}| \geq \delta_i$, μ -almost everywhere on A_j . But also, as A_j is an atom, H_{ij} has a constant sign μ -almost everywhere on A_j and so $|\int_{A_j} H_{ij} d\mu| \geq \delta_i \mu(A_j)$, contradicting (4). Therefore, $\mu(B_{ij}) = 0$ and hence $|H_{ij}| < \delta_i$, μ -almost everywhere in A_j , as claimed. We conclude that $\|\sum_{s=1}^k \rho_s [(\sum_{l=1}^k a_{ils} x_l^{**}) - z_{is}]|_{A_j}\| \leq \delta_i$, for all $i \leq n$, $j \leq q$, and all choices of signs $(\rho_s)_{s=1}^k$. Combining with (3) we deduce that $\|\sum_{s=1}^k \rho_s [(\sum_{l=1}^k a_{ils} x_l^{**}) - z_{is}]\| \leq \delta_i$, for all $i \leq n$ and all choices of signs $(\rho_s)_{s=1}^k$.

We next set $W = [\{x_l^{**} : l \leq k\} \cup \{z_{is} : i \leq n, s \leq k\}]$ and choose $0 < \delta < \epsilon$. Theorem 2.1 yields a linear operator $U : W \rightarrow X$, $\|U\| \leq 1 + \delta$, so that $U|_{X \cap W} = id_{X \cap W}$ and $g(f_j) = f_j(Ug)$, for all $g \in W$ and $j \leq q$. Setting $x_l = Ux_l^{**}/(1 + \delta)$, $l \leq k$, we obtain that for every choice of signs

ρ_1, \dots, ρ_k and all $i \leq n$,

$$\begin{aligned}
& \left\| \sum_{s=1}^k \rho_s \left[\left(\sum_{l=1}^k a_{ils} x_l \right) - z_{is} \right] \right\| = \left\| \sum_{s=1}^k \rho_s \left[\left(\sum_{l=1}^k a_{ils} U x_l^{**} / (1 + \delta) \right) - U z_{is} \right] \right\| \\
& \leq (1 + \delta) \left\| \sum_{s=1}^k \rho_s \left[\left(\sum_{l=1}^k a_{ils} x_l^{**} / (1 + \delta) \right) - z_{is} \right] \right\| \\
& \leq \left\| \sum_{s=1}^k \rho_s \left[\left(\sum_{l=1}^k a_{ils} x_l^{**} \right) - (1 + \delta) z_{is} \right] \right\| \\
& \leq \left\| \sum_{s=1}^k \rho_s \left[\left(\sum_{l=1}^k a_{ils} x_l^{**} \right) - z_{is} \right] \right\| + \delta \left\| \sum_{s=1}^k \rho_s z_{is} \right\| \leq \delta_i + \delta, \text{ as } \|T_i\| \leq 1.
\end{aligned}$$

Thus $\sum_{s=1}^k |x^*(\sum_{l=1}^k a_{ils} x_l) - x^*(z_{is})| \leq \delta_i + \delta$, for all $x^* \in B_{X^*}$. If we define $S: X^* \rightarrow V$ by $Sx^* = \sum_{l=1}^k x^*(x_l) e_l$, we see that S is w^* -continuous and $\|S\| \leq 1$. Indeed, for the latter assertion we observe that for every choice of signs ρ_1, \dots, ρ_k , $\|\sum_{l=1}^k \rho_l x_l^{**}\| \leq 1$, by the definition of the sequence $(x_l^{**})_{l=1}^k$ and the fact that $\|v_j\| \leq 1$ for $j \leq q$, and $\|T_n\| \leq 1$. It follows now, by the choice of U , that $\|\sum_{l=1}^k \rho_l x_l\| \leq 1$ for every choice of signs ρ_1, \dots, ρ_k , and therefore $\|S\| \leq 1$.

We deduce that $\|R_i S x^* - T_i x^*\| \leq \delta_i + \delta$ for all $x^* \in B_{X^*}$ and every $i \leq n$. Hence $\|R_i S - T_i\| < \delta_i + \epsilon$ for all $i \leq n$. Finally,

$$\begin{aligned}
\|S f_j - v_j\| &= \sum_{l=1}^k |f_j(x_l) - v_{jl}| \\
&= \sum_{l=1}^k |(1 + \delta)^{-1} f_j(U x_l^{**}) - v_{jl}| = \sum_{l=1}^k |(1 + \delta)^{-1} x_l^{**}(f_j) - v_{jl}| \\
&= \delta(1 + \delta)^{-1} \|v_j\| < \epsilon.
\end{aligned}$$

The proof of the proposition is now complete. \square

Proof of Theorem 3.2. We first choose $\theta_i < \delta_i$ so that $\|R_i T_n - T_i\| < \theta_i$ and $\|R_i v_j - T_i f_j\| < \theta_i$ for all $j \leq q$ and $i \leq n$. We then choose $0 < \epsilon_0 < (\delta_i - \theta_i)/4$ for all $i \leq n$, and a sequence (ϵ_m) of positive scalars such that $\sum_{m=1}^\infty \epsilon_m < \epsilon_0$.

Proposition 3.3 yields a w^* -continuous linear operator $S_0: X^* \rightarrow V$, $\|S_0\| \leq 1$, such that $\|R_i S_0 - T_i\| < \theta_i + \epsilon_0$ for $i \leq n$, and $\|S_0 f_j - v_j\| < \epsilon_0$, for $j \leq q$.

We next apply Proposition 3.3 for “ n ” = 1, “ R_1 ” = id_V , “ T_1 ” = S_0 , “ δ_1 ” = ϵ_0 and “ ϵ ” = ϵ_1 , to obtain a w^* -continuous linear operator $S_1: X^* \rightarrow V$, $\|S_1\| \leq 1$, so that $\|S_0 - S_1\| < \epsilon_0 + \epsilon_1$ and $\|S_1 f_j - v_j\| < \epsilon_1$, for $j \leq q$. Continuing in this fashion we construct w^* -continuous linear operators

$S_m: X^* \rightarrow V$ with $\|S_m\| \leq 1$ and such that

$$\|S_{m-1} - S_m\| < \epsilon_{m-1} + \epsilon_m, \|S_m f_j - v_j\| < \epsilon_m, \text{ for all } j \leq q, m \in \mathbb{N}.$$

Clearly, the sequence of operators (S_m) converges in norm to a w^* -continuous linear operator $T: X^* \rightarrow V$ such that $\|T\| \leq 1$ and $Tf_j = v_j, j \leq q$. In addition to that we have

$$\|S_m - S_0\| < \epsilon_0 + 2 \sum_{i=1}^{m-1} \epsilon_i + \epsilon_m, m > 1,$$

and thus $\|T - S_0\| < 3\epsilon_0$. We conclude that

$$\begin{aligned} \|R_i T - T_i\| &\leq \|R_i(T - S_0)\| + \|R_i S_0 - T_i\| \\ &< \theta_i + 4\epsilon_0 < \delta_i, i \leq n. \end{aligned}$$

□

Corollary 3.4. *Let X be an L_1 -predual and let V be a subspace of X^* isometric to ℓ_1^k . Assume that there exist collections of vectors $(f_j)_{j=1}^q \subset \text{ext}(B_{X^*})$ and $(v_j)_{j=1}^q \subset B_V$, with $(f_j)_{j=1}^q$ linearly independent, so that every linear operator $R: X^* \rightarrow X^*$ satisfying $Rf_j = v_j$ for all $j \leq q$, also satisfies $R|_V = \text{id}_V$. Then there exists a w^* -continuous contractive projection $P: X^* \rightarrow V$, such that $Pf_j = v_j$ for all $j \leq q$.*

Proof. We apply Theorem 3.2 for $n = 1, R_1 = \text{id}_V, T_1 = \mathbf{0}$ and $\delta_1 > 1$, to obtain a w^* -continuous linear operator $P: X^* \rightarrow V, \|P\| \leq 1$, so that $Pf_j = v_j$ for all $j \leq q$. Our assumptions yield that P is the desired projection. □

Remark . *We note that Lemma 3.1 and Corollary 3.2 of [10] yield for every $\epsilon > 0$ a w^* -continuous projection $P: X^* \rightarrow V, \|P\| \leq 1 + \epsilon$, such that $Pf_j = v_j$ for all $j \leq q$.*

4. MAIN RESULTS

In this section we present the proofs of the results mentioned in the introduction.

Proof of Theorem 1.1. Assume K is infinite and let (e_n^*) be an enumeration of K . (The argument for finite K is implicitly contained in the proof of the infinite case.) It is clear that (e_n^*) is isometrically equivalent to the usual ℓ_1 -basis. Set $Y_n = [e_1^*, \dots, e_n^*], n \in \mathbb{N}$ and let (δ_n) be a null sequence of positive scalars. We shall inductively construct w^* -continuous contractive projections $P_n: X^* \rightarrow Y_n$ such that $\|P_k P_n - P_k\| < \delta_k$, whenever $k \leq n$. P_1 is selected by applying Corollary 3.4 for the subspace Y_1 and the vectors $f_1 = v_1 = e_1^*$. Suppose $(P_i)_{i=1}^n$ have been selected so that $\|P_i P_j - P_i\| < \delta_i$ whenever $i \leq j \leq n$. Apply Theorem 3.2 for “ V ” = Y_{n+1} , “ T_i ” = P_i , “ R_i ” = $P_i|_{Y_{n+1}}, i \leq n$, and the collections of vectors “ $(f_j)_{j=1}^q$ ” = $(e_j^*)_{j=1}^{n+1}$, “ $(v_j)_{j=1}^q$ ” = $(e_j^*)_{j=1}^{n+1}$, in order to obtain a w^* -continuous linear operator $P_{n+1}: X^* \rightarrow Y_{n+1}, \|P_{n+1}\| \leq 1$, such that $P_{n+1}e_j^* = e_j^*$ for

all $j \leq n+1$, and $\|P_i P_{n+1} - P_i\| < \delta_i$ for all $i \leq n$. Clearly, P_{n+1} is the required projection onto Y_{n+1} . This completes the inductive construction. The assertion of the theorem now follows from Proposition 3.1. \square

Proof of Corollary 1.2. Clearly, K is linearly independent. When K is finite the assertion follows immediately from Theorem 1.1 as $[K]$ is isometric to $\ell_1^{|K|} = C(K)^*$. If K is infinite let (e_n^*) be an enumeration of K and set $Y = [(e_n^*)]$. Of course (e_n^*) is isometrically equivalent to the usual ℓ_1 -basis, and applying the Choquet representation and the Krein-Millman theorems, we infer that $\overline{\text{co}}^{w^*}(K \cup -K) = B_Y$. A classical result [12] yields that Y is w^* -closed in X^* . It is not hard to see (cf. also Lemma 2 of [2]) that Y is w^* -isometric to $C(K)^*$. The result follows from Theorem 1.1. \square

Proof of Corollary 1.3. We regard B_{X^*} in its w^* -topology and set $H = \text{ext}(B_{X^*})$. Since X is separable and X^* is non-separable, H is an uncountable G_δ -subset of B_{X^*} . It follows that H is an uncountable Polish space in its relative w^* -topology. We will show that H contains a w^* -compact subset L homeomorphic to the Cantor set Δ , such that $L \cap (-L) = \emptyset$. Indeed, let $\sigma: H \rightarrow H$ denote negation ($\sigma h = -h$). Then σ is a fixed-point free homeomorphism on the uncountable Polish space H and therefore there exists an uncountable relatively open subset G of H such that $G \cap \sigma G = \emptyset$. By a classical result G contains a compact subset L homeomorphic to the Cantor set which clearly satisfies $L \cap (-L) = \emptyset$.

Of course L contains homeomorphs of all countable compact metric spaces. Corollary 1.2 now yields that X contains subspaces isometric to $C(K)$, for every countable compact metric space K . Because X is separable, a result of Bourgain (Proposition 9 of [4]) implies that X contains a subspace isometric to $C(\Delta)$. The existence of a contractively complemented subspace of X isometric to $C(\Delta)$ now follows from a result of Pelczynski [22]. \square

Proof of Theorem 1.4. We first choose an infinite w^* -convergent subsequence $(e_m^*)_{m \in M}$ of (e_n^*) and set $x_0^* = w^* - \lim_{m \in M} e_m^*$. Clearly, $\|x_0^*\| \leq 1$. If $x_0^* = \mathbf{0}$ then $Z = [(e_m^*)_{m \in M}]$ is w^* -closed in X^* by Lemma 1 of [2]. We deduce from Theorem 1.1 that Z is the range of a w^* -continuous contractive projection in X^* . It is easy to see that if (r_n) is an enumeration of M then the subspace $Y = [(e_{r_{2n}}^* - e_{r_{2n-1}}^*)]$ is the range of a w^* -continuous contractive projection in Z . Hence by composing the projections previously obtained we see that the assertion of the theorem holds in this case.

We shall next deal with the case of $x_0^* \neq \mathbf{0}$. Suppose that $x_0^* = \sum_{j=1}^{\infty} a_j e_j^*$ and choose a sequence of positive scalars (ϵ_i) such that $\sum_{i=1}^{\infty} \epsilon_i < 1$. Choose also $n_1 \in \mathbb{N}$ so that $\sum_{j > n_1} |a_j| < \epsilon_1 \sum_{j \leq n_1} |a_j|$. We shall inductively construct increasing sequences $(m_k)_{k=1}^{\infty} \subset M$ and $(n_k)_{k=1}^{\infty} \subset \mathbb{N}$ with $n_k < m_{2k-1} < m_{2k} < n_{k+1}$, and w^* -continuous contractive projections $P_k: X^* \rightarrow Y_k$, where $Y_k = [u_i^* : i \leq k]$ and $u_k^* = (e_{m_{2k}}^* - e_{m_{2k-1}}^*)/2$, $k \in \mathbb{N}$, so that the

following conditions are fulfilled:

$$(5) \quad \sum_{j > n_i} |a_j| < \epsilon_i \sum_{j \leq n_1} |a_j|, \quad i \in \mathbb{N}.$$

$$(6) \quad P_1 e_j^* = \mathbf{0}, \quad j \leq n_1, \quad \text{while } \|P_i e_j^*\| < \sum_{l < i} \epsilon_l, \quad j \leq n_1, \quad i \geq 2.$$

$$(7) \quad P_i \left(\sum_{j \leq n_i} a_j e_j^* \right) = \mathbf{0}, \quad i \in \mathbb{N}.$$

$$(8) \quad P_i e_{m_{2j}}^* = u_j^*, \quad P_i e_{m_{2j-1}}^* = -u_j^*, \quad j \leq i, \quad i \in \mathbb{N}.$$

$$(9) \quad \|P_i P_j - P_i\| < \sum_{l=i}^{j-1} \epsilon_l, \quad i < j \text{ in } \mathbb{N}.$$

$$(10) \quad \|P_i e_{m_j}^*\| < \epsilon_i, \quad j \in \{2l-1, 2l\}, \quad i < l \text{ in } \mathbb{N}.$$

Once this is accomplished, condition (9) will enable us to apply Proposition 3.1 and deduce that $Y = [(u_k^*)]$ is the range of a w^* -continuous projection in X^* . Note that Y is w^* -closed in X^* by Lemma 1 of [2] as (u_k^*) is isometrically equivalent to the usual ℓ_1 -basis.

We first choose $m_1 < m_2$ in M with $m_1 > n_1$, and apply Corollary 3.4 for “ V ” = Y_1 , “ q ” = $n_1 + 2$, “ f_j ” = e_j^* ($j \leq n_1$), “ f_{q-1} ” = $e_{m_1}^*$, “ f_q ” = $e_{m_2}^*$, and “ v_j ” = $\mathbf{0}$ ($j \leq n_1$), “ v_{q-1} ” = $-u_1^*$, “ v_q ” = u_1^* . We obtain a w^* -continuous contractive projection $P_1: X^* \rightarrow Y_1$ such that $P_1 e_j^* = \mathbf{0}$ for all $j \leq n_1$.

Suppose that we have constructed $(n_i)_{i=1}^k$, $(m_i)_{i=1}^{2k}$ and $(P_i)_{i=1}^k$ so that conditions (5)-(10) are satisfied. We next choose $n_{k+1} > m_{2k}$ so that (5) is satisfied for $i = k+1$. By (5) and (7) of the induction hypothesis we infer that $\|P_i x_0^*\| < \epsilon_i$, for $i \leq k$. We can therefore choose $m_{2k+1} < m_{2k+2}$ in M with $m_{2k+1} > n_{k+1}$, such that $\|P_i e_{m_{2k+1}}^*\| < \epsilon_i$ and $\|P_i e_{m_{2k+2}}^*\| < \epsilon_i$ for all $i \leq k$. Hence (10) is satisfied for $l = k+1$.

We next put $q = n_{k+1} + 2$ and set $f_j = e_j^*$, for $j \leq n_{k+1}$, $f_{q-1} = e_{m_{2k+1}}^*$ and $f_q = e_{m_{2k+2}}^*$. We claim that there exist vectors $(v_j)_{j=1}^q$ in $B_{Y_{k+1}}$ so that

$$(11) \quad \|v_j\| < \sum_{l=1}^k \epsilon_l, \quad j \leq n_1.$$

$$(12) \quad v_j = P_k e_j^*, \quad j \in (n_1, n_{k+1}], \quad v_{q-1} = -u_{k+1}^*, \quad v_q = u_{k+1}^*.$$

$$(13) \quad \sum_{j \leq n_{k+1}} a_j v_j = \mathbf{0}.$$

$$(14) \quad \|P_i f_j - P_i v_j\| < \sum_{l=i}^k \epsilon_l, \quad i \leq k, \quad j \leq q.$$

Having achieved this and taking in account (9) of the induction hypothesis, we employ Theorem 3.2 for “ V ” = Y_{k+1} , “ n ” = k , “ δ_i ” = $\sum_{l=i}^k \epsilon_l$, “ T_i ” = P_i , “ R_i ” = $P_i|_{Y_{k+1}}$ ($i \leq k$), and the collections of vectors $(f_j)_{j=1}^q$, $(v_j)_{j=1}^q$

described above, to find a w^* -continuous linear operator $P_{k+1}: X^* \rightarrow Y_{k+1}$, $\|P_{k+1}\| \leq 1$, such that $\|P_i P_{k+1} - P_i\| < \sum_{l=i}^k \epsilon_l$, for all $i \leq k$, and $P_{k+1} f_j = v_j$ for all $j \leq q$. It is easy to verify that P_{k+1} is a projection onto Y_{k+1} so that $(n_i)_{i=1}^{k+1}$, $(m_i)_{i=1}^{2k+2}$ and $(P_i)_{i=1}^{k+1}$ satisfy conditions (5)-(10).

The collection $(v_j)_{j=n_1+1}^q$ is explicitly defined in (12). It remains to define $(v_j)_{j=1}^{n_1}$. We first choose scalars (b_{il}) , where $i \leq k$ and $l \in (n_k, n_{k+1}]$, such that $P_k e_l^* = \sum_{i=1}^k b_{il} u_i^*$. Note that $\sum_{i=1}^k |b_{il}| \leq 1$ for every $l \in (n_k, n_{k+1}]$ since $\|P_k\| = 1$. We also define scalars $(\rho_i)_{i=1}^k$ by

$$\rho_i = \left(- \sum_{l \in (n_k, n_{k+1}]} a_l b_{il} \right) / \sum_{j \leq n_1} |a_j|,$$

and set $\rho_{ij} = sg(a_j) \rho_i$ for $i \leq k$ and $j \leq n_1$. Observe that $\sum_{i=1}^k |\rho_i| < \epsilon_k$, by (5) of the induction hypothesis. We now set

$$v_j = P_k e_j^* + \sum_{i=1}^k \rho_{ij} u_i^*, \quad j \leq n_1.$$

It follows now by (6) of the induction hypothesis that (11) is satisfied. To establish (13) we have

$$\begin{aligned} \sum_{j \leq n_{k+1}} a_j v_j &= \sum_{j \leq n_1} a_j v_j + \sum_{j \in (n_1, n_{k+1}]} a_j P_k e_j^* \\ &= \sum_{j \leq n_1} a_j \sum_{i=1}^k \rho_{ij} u_i^* + \sum_{j \leq n_{k+1}} a_j P_k e_j^* \\ &= \sum_{j \leq n_1} |a_j| \sum_{i=1}^k \rho_i u_i^* + \sum_{j \leq n_k} a_j P_k e_j^* + \sum_{j \in (n_k, n_{k+1}]} a_j P_k e_j^* \\ &= - \sum_{i=1}^k \sum_{l \in (n_k, n_{k+1}]} a_l b_{il} u_i^* + \sum_{j \in (n_k, n_{k+1}]} a_j P_k e_j^*, \text{ by (7)} \\ &= - \sum_{l \in (n_k, n_{k+1}]} a_l \sum_{i=1}^k b_{il} u_i^* + \sum_{j \in (n_k, n_{k+1}]} a_j P_k e_j^* \\ &= - \sum_{l \in (n_k, n_{k+1}]} a_l P_k e_l^* + \sum_{j \in (n_k, n_{k+1}]} a_j P_k e_j^* = \mathbf{0}. \end{aligned}$$

Finally, we show that (14) holds. Indeed, when $j \in \{q-1, q\}$, this is a consequence of the choice of m_{2k+1} and m_{2k+2} . When $j \in (n_1, n_{k+1}]$ the assertion follows from (9) of the induction hypothesis. When $j \leq n_1$ it follows from (9) of the induction hypothesis and the fact that $\sum_{i=1}^k |\rho_i| < \epsilon_k$. \square

Proof of Corollary 1.5. If X^* is non-separable the assertion follows from Corollary 1.3. If X^* is separable, then $\text{ext}(B_{X^*})$ is countable and X^* is isometric to ℓ_1 . Let (e_n^*) be a basis for X^* isometrically equivalent to the usual ℓ_1 -basis, and choose a w^* -convergent subsequence $(e_{m_n}^*)$ of (e_n^*) according to Theorem 1.4. Let $Y = [(e_{m_{2n}}^* - e_{m_{2n-1}}^*)]$. Then it is easy to see that Y is w^* -isometric to c_0^* . The result follows from Theorem 1.4. \square

Our last corollary is a special case of the structural result for separable L_1 -preduals established in [20] and [15].

Corollary 4.1. *Suppose that X^* is isometric to ℓ_1 . Then there exists a sequence (E_n) of finite-dimensional subspaces of X such that $E_n \subset E_{n+1}$ for all n , each E_n is isometric to ℓ_∞^n , and $\cup_{n=1}^\infty E_n$ is dense in X .*

Proof. Let (e_n^*) be a basis for X^* isometrically equivalent to the usual ℓ_1 -basis. Let $Y_n = [e_i^* : i \leq n]$, $n \in \mathbb{N}$, and let (δ_n) be a sequence of positive scalars such that $\sum_{n=1}^\infty \delta_n < \infty$. The argument in the proof of Theorem 1.1 now yields a sequence (P_n) of w^* -continuous contractive projections in X^* with $\text{Im} P_n = Y_n$ and such that $\|P_k P_n - P_k\| < \delta_k$ whenever $k \leq n$. Given $k \leq n$ in \mathbb{N} , we set $Q_k^n = P_k \cdots P_n$. Clearly Q_k^n is a w^* -continuous contractive projection onto Y_k . Moreover, our assumptions yield that $\|Q_k^n - Q_k^{n+1}\| < \delta_n$, for all $n \geq k$ and thus the sequence of operators $(Q_k^n)_{n \geq k}$ converges in norm to a w^* -continuous contractive projection Q_k from X^* onto Y_k . It is easily seen that $Q_k^m Q_l^n = Q_k^n$, whenever $k \leq l \leq m \leq n$ and hence $Q_k Q_l = Q_k$ when $k \leq l$.

We now let $E_n = Q_n^* Y_n^*$. E_n is naturally identified to a subspace of X as Q_n is w^* -continuous, and of course it is isometric to ℓ_∞^n for all $n \in \mathbb{N}$. Since $Q_n Q_{n+1} = Q_n$ we deduce that $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}$. It is also easily verified that Q_n^* acts as a contractive projection from X onto E_n . Rainwater's theorem now yields that $\lim_n Q_n^* x = x$, weakly, for all $x \in X$ and thus $\cup_{n=1}^\infty E_n$ is dense in X . \square

- Remark .**
1. *We note that the proof of Corollary 1.5 that appears in [27] makes use of the structural result established in [20] and [15].*
 2. *Theorem 1.4 is no longer valid if we consider isomorphic ℓ_1 -preduals. Indeed, it is shown in [5] that there exist isomorphic ℓ_1 -preduals which do not contain isomorphic copies of c_0 .*
 3. *According to a result of Fonf [7], every Banach space X such that $\text{ext}(B_{X^*})$ is countable contains a subspace isomorphic to c_0 .*

It was shown in [11] that every separable L_1 -predual is isometric to a quotient of $C(\Delta)$. It is an open problem [2] whether every ℓ_1 -predual is isomorphic, or even almost isometric, to a quotient of $C(K)$ for some countable compact metric space K .

Question: Suppose X is an ℓ_1 -predual such that for some $\epsilon > 0$ and some countable ordinal α , the ϵ -Szlenk index of X [25] exceeds ω^α . Does X contain a contractively complemented subspace isometric to $C_0(\omega^{\omega^\alpha})$? Does X contain a subspace isomorphic to $C(\omega^{\omega^\alpha})$?

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